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Vertical Transmission of HIV-HBV Co-infection with Liquor Habit and Vaccination

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Abstract

In this paper, the transmission of HIV-HBV co-infection is carried out. The individuals who are infected with both diseases HIV and HBV simultaneously, are said to be HIV-HBV co-infected. These infected individuals have high risk of liver failure. It is the main cause for serious liver complications like cirrhosis and liver cancer at younger age. A deterministic model is considered with liquor habit in men and vaccination to new-borns and carrier mother. Carrier class results in the vertical transmission. In this paper, the transmission dynamics of the model is analyzed. The total population is divided in to twenty eight class viz. Susceptible, HBV Vaccinated, HBV-infected female, HBV-carrier female, HBV infected alcoholic male, HBV carrier alcoholic male, HBV infected non-alcoholic male, HBV carrier non-alcoholic male, HBV recovered class, pre-AIDS female, AIDS female, pre-AIDS-HBV co-infected female, AIDS-HBV co-infected female, pre-AIDS-HBV carrier female, AIDS-HBV carrier female, pre-AIDS alcoholic male, AIDS alcoholic male, pre-AIDS non-alcoholic male, AIDS non-alcoholic male, pre-AIDS-HBV co-infected alcoholic male, pre-AIDS-HBV co-infected non-alcoholic male, pre-AIDS-HBV carrier alcoholic male, pre-AIDS-HBV carrier non-alcoholic male, AIDS-HBV co-infected alcoholic male, AIDS-HBV co-infected non-alcoholic male, AIDS-HBV carrier non-alcoholic male, HIV infected -HBV recovered classes. The basic reproduction numbers for HIV, for HBV and for HIV-HBV are found using next generation matrix. Local and global stability of HIV-HBV disease free equilibrium is worked out. Model is validated with the numerical simulation.

Keywords: HIV-HBV co-infection; vertical transmission; liquor habits; vaccination; basic reproduction number; disease free equilibrium; local stability; global stability.

1 Introduction

1.1 Hepatitis B Virus (HBV)

A viral infection Hepatitis B attacks on the liver and can cause both acute and chronic disease as per the report of [8]. Hepatitis B (HBV) is a major threat globally. The virus can be transmitted through contact with the body fluids of an infected person. [1] reported that in the Indian subcontinent, approximately 2–5% of the general population is chronically infected and worldwide, an estimated 370 million people are chronically infected with hepatitis B. Worldwide 7,80,000 people die each year due to acute HBV. [2] and [3] established that the sexual transmission plays vital role in spreading HBV. Vaccination gives immunity against HBV. Pregnant female are at high risk of HBV. The immunization of pregnant female/carrier female is preventive measure in medical science.

1.2 Human Immunodeficiency Virus (HIV)

The transmission of HIV/AIDS depends on many factors. It is observed that high rate of HIV/AIDS is in a class of low literate and high poverty. The [7] reported that about 20% of the children infected with HIV catch up AIDS in the neonatal age and die within first four years. The remaining 80% of infected children develops AIDS in due course of life. Worldwide, an estimated 40 million people are infected with HIV as per [7]. [6] observed that starting treatment earlier can control the disease spread.

1.3 HIV-HBV Co-infection

A person who is infected with both the Hepatitis B and the HIV viruses is said to have a HBV/HIV co-infection. [1] estimated that 2 to 4 million have chronic HBV co-infection amongst HIV infected persons. People co-infected with Hepatitis B and HIV both are 14 to 17 times more likely to die than those with hepatitis B alone. [8] reported that the co-infection of HIV and HBV also accelerates liver scarring. Some medicines which are used to treat HIV are toxic to the liver and it makes matter worse. It is critical to understand the study of co-infectious disease and to understand how diseases are related, what type of effective treatment and prevention should be taken. [4] formatted mathematical models which provide insight into the disease dynamics, and predicts effective control measures.

1.4 Proposed Vertical transmission

In this study, mathematical model for HIV-HBV co-infection with vaccination and alcoholic habits is formulated and analysed. The epidemiological and biological features of both diseases are incorporated. The next generation matrix method is used to find basic reproduction number for co-infection suggested by [5].

The paper is organised as follows. In section 2, we develop mathematical model through transfer diagram by defining model parameters. In section 3, stability analysis of the system is discussed and the model is interpreted using numerical simulations In section 4. In section 5, conclusions are summarized.

2 Mathematical Model

To study, the spread of HIV-HBV co-infection under vaccination and liquor habits, a non-linear mathematical system of differential equations is formulated. Total population is divided in to twenty eight compartments, viz. Susceptible, HBV Vaccinated, HBV-infected female, HBV-carrier female, HBV infected alcoholic male, HBV carrier alcoholic male, HBV infected non-alcoholic male, HBV carrier non-alcoholic male, HBV recovered class, pre-AIDS female, AIDS female, pre-AIDS-HBV co-infected female, AIDS-HBV co-infected female, pre-AIDS-HBV carrier female, AIDS-HBV carrier female, pre-AIDS alcoholic male, AIDS alcoholic male, pre-AIDS non-alcoholic male, AIDS non-alcoholic male, pre-AIDS-HBV co-infected alcoholic male, pre-AIDS-HBV co-infected non-alcoholic male, pre-AIDS-HBV carrier alcoholic male, pre-AIDS-HBV carrier non-alcoholic male, AIDS-HBV co-infected alcoholic male, AIDS-HBV co-infected non-alcoholic male, AIDS-HBV carrier non-alcoholic male, HIV infected -HBV recovered classes.

Mathematical model is derived with following notations.

Table 1: Notations with model parameter values.

Notation	Details	Parametric Values
$N(t)$	Total Population at time t	10000
$S(t)$	Number of susceptible HBV and HIV individuals at time t	100
$V_B(t)$	Number of vaccinated (HBV) individuals at time t	98
$B_{fI}(t)$	Number of HBV infected female individuals at time t	5
$B_{fC}(t)$	Number of HBV carrier female individuals at time t	7
$P_f(t)$	Number of Pre-AIDS female individuals at time t	8
$A_f(t)$	Number of AIDS female individuals at time t	5
$PB_{fI}(t)$	Number of Pre-AIDS and HBV co-infected female individuals at time t	10
$AB_{fI}(t)$	Number of AIDS and HBV co-infected female individuals at time t	8
$PB_{fC}(t)$	Number of Pre-AIDS infected HBV carrier female individuals at time t	10
$AB_{fC}(t)$	Number of AIDS infected HBV carrier female individuals at time t	8
$B_{maI}(t)$	Number of HBV infected alcoholic male individuals at time t	10
$B_{maC}(t)$	Number of HBV carrier alcoholic male individuals at time t	10
$P_{ma}(t)$	Number of Pre-AIDS alcoholic male individuals at time t	8
$A_{ma}(t)$	Number of AIDS alcoholic male individuals at time t	8
$PB_{maI}(t)$	Number of Pre-AIDS and HBV co-infected alcoholic male individuals at time t	15

Notation	Details	Parametric Values
$AB_{maI}(t)$	Number of AIDS and HBV co-infected alcoholic male individuals at time t	8
$PB_{maC}(t)$	Number of Pre-AIDS infected HBV carrier alcoholic male individuals at time t	15
$AB_{maC}(t)$	Number of -AIDS infected HBV carrier alcoholic male individuals at time t	8
$B_{mnaI}(t)$	Number of HBV infected non-alcoholic male individuals at time t	5
$B_{mnaC}(t)$	Number of HBV carrier non-alcoholic male individuals at time t	5
$P_{mna}(t)$	Number of Pre-AIDS non-alcoholic male individuals at time t	8
$A_{mna}(t)$	Number of AIDS non-alcoholic male individuals at time t	5
$PB_{mnaI}(t)$	Number of Pre-AIDS and HBV co-infected non-alcoholic male individuals at time t	12
$AB_{mnaI}(t)$	Number of AIDS and HBV co-infected non-alcoholic male individuals at time t	5
$PB_{mnaC}(t)$	Number of Pre-AIDS infected HBV carrier non-alcoholic male individuals at time t	12
$AB_{mnaC}(t)$	Number of AIDS infected HBV carrier non-alcoholic male individuals at time t	5
$R_B(t)$	HBV recovered individuals at time t	3
$R_{PAB}(t)$	HIV/AIDS infected but HBV recovered individuals at time t	8
B	New recruitments	200
β	Recruitment rate of HBV non-vaccinated individuals at birth time	0.02
ω	Proportion of the new birth without successful vaccination	0.05
π_{Bma}	The rate of waning vaccine-induced immunity in male alcoholic	0.03
π_{Bmna}	The rate of waning vaccine-induced immunity in male non-alcoholic	0.001
π_{Bf}	The rate of waning vaccine-induced immunity in female	0.001
δ	Vaccination rate	0.98
γ_{f1}	Rate of pre-AIDS infected female moving for AIDS infected female class	0.5
γ_{f2}	Rate of HBV, pre-AIDS co-infected female moving for HBV, AIDS co- infected female class	0.6
γ_{f3}	Rate of HBV, pre-AIDS co-infected carrier female moving for HBV, AIDS co-infected carrier female class	0.8
γ_{ma}	Rate of pre-AIDS infected alcoholic male moving for AIDS infected alcoholic male class	0.8
γ_{ma1}	Rate of HBV, pre-AIDS co-infected male alcoholic moving for HBV, AIDS co- infected male alcoholic class	0.8
γ_{ma2}	Rate of HBV, pre-AIDS co-infected carrier male alcoholic moving for HBV, AIDS co-infected carrier male alcoholic class	0.8

Notation	Details	Parametric Values
γ_{mna}	Rate of pre-AIDS infected non-alcoholic male moving for AIDS infected non-alcoholic male class	0.7
γ_{mna1}	Rate of HBV, pre-AIDS co-infected non-alcoholic male moving for HBV, AIDS co- infected non-alcoholic male class	0.7
γ_{mna2}	Rate of HBV, pre-AIDS co-infected non-alcoholic carrier male moving for HBV, AIDS co-infected non-alcoholic male carrier class	0.5
μ	Natural death rate	0.4
α_1	Disease induced death rate because of HBV infection	0.05
α_2	Disease induced death rate because of AIDS	0.1
τ_1	Probability of acquiring HBV of individuals in P_f	0.3
τ_2	Probability of acquiring HBV of individuals in A_f	0.6
τ_3	Probability of acquiring HBV of individuals in P_{ma}	0.3
τ_4	Probability of acquiring HBV of individuals in P_{mna}	0.3
τ_5	Probability of acquiring HBV of individuals in A_{ma}	0.6
τ_6	Probability of acquiring HBV of individuals in A_{mna}	0.6
ψ_1	Probability of acquiring HIV of individuals in B_{fI}	0.4
ψ_2	Probability of acquiring HIV of individuals in B_{fC}	0.5
ψ_3	Probability of acquiring HIV of individuals in B_{maI}	0.4
ψ_4	Probability of acquiring HIV of individuals in B_{mnaI}	0.4
ψ_5	Probability of acquiring HIV of individuals in B_{maC}	0.75
ψ_6	Probability of acquiring HIV of individuals in B_{mnaC}	0.3
ψ	Probability of acquiring HIV of individuals in R_B	0.1
q_f	Probability of HBV infected individual female acquires female carrier class	0.05
q_{PBf}	Probability of HIV-HBV co-infected individual female acquires HIV-HBV female carrier class	0.001
q_{ABf}	Probability of AIDS-HBV co-infected individual female acquires AIDS-HBV female carrier class	0.001
q_{ma}	Probability of HBV infected individual alcoholic male acquires alcoholic male carrier class	0.07
q_{mna}	Probability of HBV infected individual non-alcoholic male acquires non-alcoholic male carrier class	0.05
q_{PBma}	Probability of HIV-HBV co-infected individual alcoholic male acquires HIV-HBV alcoholic male carrier class	0.08
q_{ABma}	Probability of AIDS-HBV co-infected individual alcoholic male acquires AIDS-HBV alcoholic male carrier class	0.08
q_{PBmna}	Probability of HIV-HBV co-infected individual non-alcoholic male acquires HIV-HBV alcoholic male carrier class	0.04
q_{ABmna}	Probability of AIDS-HBV co-infected individual non-alcoholic male acquires AIDS-HBV alcoholic male carrier class	0.04
r_f	Number of HBV infected individual female acquires female carrier class	8
r_{PBf}	Number of HIV-HBV co-infected individual female acquires HIV-HBV female carrier class	2

Notation	Details	Parametric Values
r_{ABf}	Number of AIDS-HBV co-infected individual female acquires AIDS-HBV female carrier class	2
r_{ma}	Number of HBV infected individual alcoholic male acquires alcoholic male carrier class	5
r_{mna}	Number of HBV infected individual non-alcoholic male acquires non-alcoholic male carrier class	3
r_{PBma}	Number of HIV-HBV co-infected individual alcoholic male acquires HIV-HBV alcoholic male carrier class	6
r_{ABma}	Number of AIDS-HBV co-infected individual alcoholic male acquires AIDS-HBV alcoholic male carrier class	6
r_{PBma}	Number of HIV-HBV co-infected individual alcoholic male acquires HIV-HBV alcoholic male carrier class	6
r_{ABma}	Number of AIDS-HBV co-infected individual alcoholic male acquires AIDS-HBV alcoholic male carrier class	6
r_{PBmna}	Number of HIV-HBV co-infected individual non-alcoholic male acquires HIV-HBV alcoholic male carrier class	2
r_{ABmna}	Number of AIDS-HBV co-infected individual non-alcoholic male acquires AIDS-HBV alcoholic male carrier class	2
k_f	Rate of HBV female carrier moves towards HBV recovered R_B class	0.7
k_{ma}	Rate of HBV alcoholic male carrier moves towards HBV recovered R_B class	0.3
k_{mna}	Rate of HBV non-alcoholic male carrier moves towards HBV recovered R_B class	0.8
k_{PBf}	Rate of HIV-HBV female carrier moves towards HBV recovered R_{PAB} class	0.01
k_{ABf}	Rate of AIDS-HBV female carrier moves towards HBV recovered R_{PAB} class	0.01
k_{PBma}	Rate of HIV-HBV alcoholic male carrier moves towards HBV recovered R_{PAB} class	0.01
k_{ABma}	Rate of AIDS-HBV alcoholic male carrier moves towards HBV recovered R_{PAB} class	0.01
k_{PBmna}	Rate of HIV-HBV non-alcoholic male carrier moves towards HBV recovered R_{PAB} class	0.05
k_{ABmna}	Rate of AIDS-HBV non-alcoholic male carrier moves towards HBV recovered R_{PAB} class	0.05
v_1	Proportion of Perinatal infected new borns by carrier mother in B_{fC}	0.12
v_2	Proportion of Perinatal infected new borns by carrier mother in B_{mnaC}	0.01
v_3	Proportion of Perinatal infected new borns by carrier mother in AB_{fC}	0.001
v_4	Proportion of Perinatal infected new borns by carrier mother in AB_{mnaC}	0.002

Notation	Details	Parametric Values
v_5	Proportion of Perinatal infected new borns by carrier mother in PB_{fC}	0.001
v_6	Proportion of Perinatal infected new borns by carrier mother in PB_{mnaC}	0.001
β_{Bf}	The effective contact rate for Sexual transmission of HBV amongst female	0.0004
ω_1	The relative infectiousness related to β_{Bf} of individuals in the PB_{maI} and PB_{mnaI}	0.01
ω_2	The relative infectiousness related to β_{Bf} of individuals in the AB_{maI} and AB_{mnaI}	0.01
ω_3	The relative infectiousness related to β_{Bf} of individuals in the B_{maC} and B_{mnaC}	0.03
ω_4	The relative infectiousness related to β_{Bf} of individuals in the PB_{maC} and PB_{mnaC}	0.04
ω_5	The relative infectiousness related to β_{Bf} of individuals in the AB_{maC} and AB_{mnaC}	0.02
β_{Bma}	The effective contact rate for Sexual transmission of HBV amongst alcoholic male	0.0001
ω_6	The relative infectiousness related to β_{Bma} of individuals in the PB_{fI}	0.01
ω_7	The relative infectiousness related to β_{Bma} of individuals in the AB_{fI}	0.01
ω_8	The relative infectiousness related to β_{Bma} of individuals in the B_{fC}	0.03
ω_9	The relative infectiousness related to β_{Bma} of individuals in the PB_{fC}	0.04
ω_{10}	The relative infectiousness related to β_{Bma} of individuals in the AB_{fC}	0.2
β_{Bmna}	The effective contact rate for Sexual transmission of HBV amongst non-alcoholic male	0.0001
ω_{11}	The relative infectiousness related to β_{Bmna} of individuals in the PB_{fI}	0.01
ω_{12}	The relative infectiousness related to β_{Bmna} of individuals in the AB_{fI}	0.01
ω_{13}	The relative infectiousness related to β_{Bmna} of individuals in the B_{fC}	0.02
ω_{14}	The relative infectiousness related to β_{Bmna} of individuals in the PB_{fC}	0.01
ω_{15}	The relative infectiousness related to β_{Bmna} of individuals in the AB_{fC}	0.01

Notation	Details	Parametric Values
β_{Pf}	The effective contact rate for Sexual transmission of HIV amongst female	0.0003
η_1	The relative infectiousness related to β_{Pf} of individuals in the A_{ma} and A_{mna}	0.03
η_2	The relative infectiousness related to β_{Pf} of individuals in the PB_{maI} and PB_{mnaI}	0.01
η_3	The relative infectiousness related to β_{Pf} of individuals in the AB_{maI} and AB_{mnaI}	0.03
η_4	The relative infectiousness related to β_{Pf} of individuals in the PB_{maC} and PB_{mnaC}	0.04
η_5	The relative infectiousness related to β_{Pf} of individuals in the AB_{maC} and AB_{mnaC}	0.02
β_{Pma}	The effective contact rate for Sexual transmission of HIV amongst alcoholic male	0.005
η_6	The relative infectiousness related to β_{Pma} of individuals in the A_f	0.001
η_7	The relative infectiousness related to β_{Pma} of individuals in the PB_{fI}	0.01
η_8	The relative infectiousness related to β_{Pma} of individuals in the AB_{fI}	0.03
η_9	The relative infectiousness related to β_{Pma} of individuals in the PB_{fC}	0.04
η_{10}	The relative infectiousness related to β_{Pma} of individuals in the AB_{fC}	0.2
β_{Pmna}	The effective contact rate for Sexual transmission of HIV amongst non-alcoholic male	0.0003
η_{11}	The relative infectiousness related to β_{Pmna} of individuals in the A_f	0.001
η_{12}	The relative infectiousness related to β_{Pmna} of individuals in the PB_{fI}	0.01
η_{13}	The relative infectiousness related to β_{Pmna} of individuals in the AB_{fI}	0.02
η_{14}	The relative infectiousness related to β_{Pmna} of individuals in the PB_{fC}	0.01
η_{15}	The relative infectiousness related to β_{Pmna} of individuals in the AB_{fC}	0.01
β_B	The effective contact rate for non-sexual transmission of HBV	0.005

Notation	Details	Parametric Values
ω_{16}	The relative infectiousness related to β_B of individuals in the B_{fC}, B_{maC}, B_{mnaC}	0.05
ω_{17}	The relative infectiousness related to β_B of individuals in the PB_{fI}, PB_{maI}, PB_{mnaI}	0.01
β_P	The effective contact rate for non-sexual transmission of HIV	0.002
η_{16}	The relative infectiousness related to β_p of individuals in the P_{fI}, P_{maI}, P_{mnaI}, AB_{fI}, AB_{maI}, AB_{mnaI}, PB_{fC}, PB_{maC}, PB_{mnaC}, AB_{fC}, AB_{maC}, AB_{mnaC}	0.16
η_{17}	The relative infectiousness related to β_p of individuals in the PB_{fI}, PB_{maI}, PB_{mnaI}, AB_{fI}, AB_{maI}, AB_{mnaI}, PB_{fC}, PB_{maC}, PB_{mnaC}, AB_{fC}, AB_{maC}, AB_{mnaC}	0.01
λ_{Bma}	Force of Sexual infection of HBV in male alcoholic	Model Parameter
λ_{Bmna}	Force of Sexual infection of HBV in male non-alcoholic	Model Parameter
λ_{Bf}	Force of Sexual infection of HBV in female	Model Parameter
λ_{Pma}	Force of Sexual infection of HIV in male alcoholic	Model Parameter
λ_{Pmna}	Force of Sexual infection of HIV in male non-alcoholic	Model Parameter
λ_{Pf}	Force of Sexual infection of HIV in female	Model Parameters
λ_B	Force of non-sexual infection of HBV	Model Parameter
λ_P	Force of non-sexual infection of HIV	Model Parameter

To prepare the model, we have consider following possibilities of the disease spread.

i) Vertical Transmission of infection to new-borns from chronic carrier mothers.

ii) Nonsexual transmission amongst total population.

iii) Heterosexual transmissions amongst adults.

To derive the model, we have assumed that the male and female population sizes are identical and after vaccination loss of immunity is negligible. It is also considered that HIV infected individuals will have liver failure as side effects of drugs used to control HIV infection.

Disease Dynamics

We assumed that the susceptible adults become HBV, HIV or HBV and HIV co- infected via sexual contacts, blood transmission and pregnancy complications, and it leads to the birth of infected new-born. A fraction of new-born are vaccinated at birth time and hence they join susceptible class at the rate of $\beta\omega(1-(v_1+v_2)B_{fC}-(v_3+v_4)AB_{fC}-(v_5+v_6)PB_{fC})$ While, remaining $\beta\omega((v_1+v_2)B_{fC})$, $\beta\omega((v_3+v_4))AB_{fC}$ and $\beta\omega((v_5+v_6))PB_{fC}$ joins different carrier classes.

After the birth vaccination is given to susceptible, who join vaccinated class at the rate δ, remaining infected susceptible joins either of the classes B_{fI}, B_{maI} , B_{mnaI}, P_f, P_{ma} and P_{mna} at rate of $(1-\delta)\lambda_{Bf}+u_1\lambda_B$, $(1-\delta)\lambda_{Bma}+u_2\lambda_B$, $(1-\delta)\lambda_{Bmna}+(1-u_1-u_2)\lambda_B$, $(1-\delta)\lambda_{Pf}+a_1\lambda_P$, $(1-\delta)\lambda_{Pma}+a_2\lambda_P$ and $(1-\delta)\lambda_{Pmna}+(1-a_1-a_2)\lambda_P$ respectively. Here λ_{Bf}, λ_{Bma}, λ_{Bmna}, λ_{Pf}, λ_{Pma}, λ_{Pmna}, λ_B, λ_P are the forces of infection transmission. Vaccinated individuals loses their immunity and joins classes B_{fI}, B_{maI}, B_{mnaI}, P_f, P_{ma} and P_{mna} at rate of $\pi_{Bf}\lambda_{Bf}$, $\pi_{Bma}\lambda_{Bma}$, $\pi_{Bmna}\lambda_{Bmna}$, λ_{Pf}, λ_{Pma} and λ_{Pmna} respectively.

HBV infected individuals in classes B_{fI}, B_{maI} and B_{mnaI} get an acute infection and moves towards HBV carrier classes B_{fC}, B_{maC} and B_{mnaC} at the rates $q_f r_f$, $q_{ma}r_{ma}$ and $q_{mna}r_{mna}$ respectively. Remaining HBV infected individuals in B_{fI}, B_{maI} and B_{mnaI} classes either move towards recovered class R_B at the rates $(1-q_f)r_f$, $(1-q_{ma})r_{ma}$ and $(1-q_{mna})r_{mna}$ respectively or gets HIV infection and move towards PB_{fI}, PB_{maI} and PB_{mnaI} classes at the rates $\psi_1(1-q_f)\lambda_{Pf}$, $\psi_3(1-q_{ma})\lambda_{Pma}$ and $\psi_4(1-q_{mna})\lambda_{Pmna}$ respectively. HBV carrier individuals in B_{fC}, B_{maC} and B_{mnaC} classes either move towards HBV recovered class R_B at rates k_f, k_{ma} and k_{mna} respectively or die out due to HBV at rate α_1 or get HIV infection and move towards PB_{fC}, PB_{maC} and PB_{mnC} classes at the rates $\psi_2(1-k_f)\lambda_{Pf}$, $\psi_5(1-k_{ma})\lambda_{Pma}$ and $\psi_6(1-k_{mna})\lambda_{Pmna}$ respectively.

HIV infected pre-AIDS individuals in classes P_f, P_{ma} and P_{mna} either get full AIDS and moves towards classes A_f, A_{ma} and A_{mna} at the rates γ_{f1}, γ_{ma} and γ_{mna} respectively or gets HBV infection and move towards PB_{fI}, PB_{maI} and PB_{mnaI} classes at the rates $\tau_1(1-\gamma_{f1})\lambda_{Bf}$, $\tau_3(1-\gamma_{ma})\lambda_{Bma}$ and $\tau_4(1-\gamma_{mna})\lambda_{Bmna}$ respectively. Pre-AIDS-HBV co-infected PB_{fI}, PB_{maI} and PB_{mnaI} either gets acute HBV infection and move towards carrier classes PB_{fC}, PB_{maC} and PB_{mnaC} at rates $q_{PBf}(1-\gamma_{f2})r_{PBf}$, $q_{PBma}(1-\gamma_{ma1})r_{PBma}$ and $q_{PBmna}(1-\gamma_{mna1})r_{PBmna}$ respectively or get full AIDS and move towards AB_{fI}, AB_{maI} and AB_{mnaI} at rates γ_{f2}, γ_{ma1} and γ_{mna1} respectively or move towards HBV recovered class R_{PAB} at rates $(1-q_{PBf})(1-\gamma_{f2})r_{PBf}$, $(1-q_{PBma})(1-\gamma_{ma1})r_{PBma}$ and $(1-q_{PBmna})(1-\gamma_{mna1})r_{PBmna}$ respectively. pre-AIDS HBV carrier individuals PB_{fC}, PB_{maC} and PB_{mnaC} either gets full AIDS and enters in to classes AB_{fC}, AB_{maC} and AB_{mnaC} at rates γ_{f3}, γ_{ma2} and γ_{mna2} respectively or move towards HBV recovered class R_{PAB} at rates k_{PBf}, k_{PBma} and k_{PBmna} respectively or die out due to HBV at the rate α_1.

AIDS individuals A_f, A_{ma} and A_{mna} either get HBV infection and move towards AB_{fI}, AB_{maI} and AB_{mnaI} classes at the rates $\tau_2\lambda_{Bf}$, $\tau_5\lambda_{Bma}$ and $\tau_6\lambda_{Bmna}$ respectively or die out due to HIV at the rate α_2. AIDS- HBV co-infected AB_{fI}, AB_{maI} and AB_{mnaI} either gets acute HBV infection and move towards carrier classes AB_{fC}, AB_{maC} and AB_{mnaC} at rates $q_{ABf}r_{ABf}$, $q_{ABma}r_{ABma}$ and $q_{ABmna}r_{ABmna}$ respectively or move towards HBV recovered class R_{PAB} at rates $(1-q_{ABf})r_{ABf}$, $(1-q_{ABma})r_{ABma}$ and $(1-q_{ABmna})r_{ABmna}$ respectively or die out due to HIV at the rate α_2. AIDS HBV carrier individuals AB_{fC}, AB_{maC} and AB_{mnaC} either move towards HBV recovered class R_{PAB} at rates k_{ABf}, k_{ABma} and k_{ABmna} respectively or die out due to HBV at rate α_1 or die out due to HIV at rate α_2. HBV recovered individuals R_B either goes to pre-AIDS or AIDS infected-HBV recovered class R_{PAB} at the rate $\psi(\lambda_{Pf}+\lambda_{Pma}+\lambda_{Pmna})$.

The transmission of disease in various compartments is shown in Figure 1.

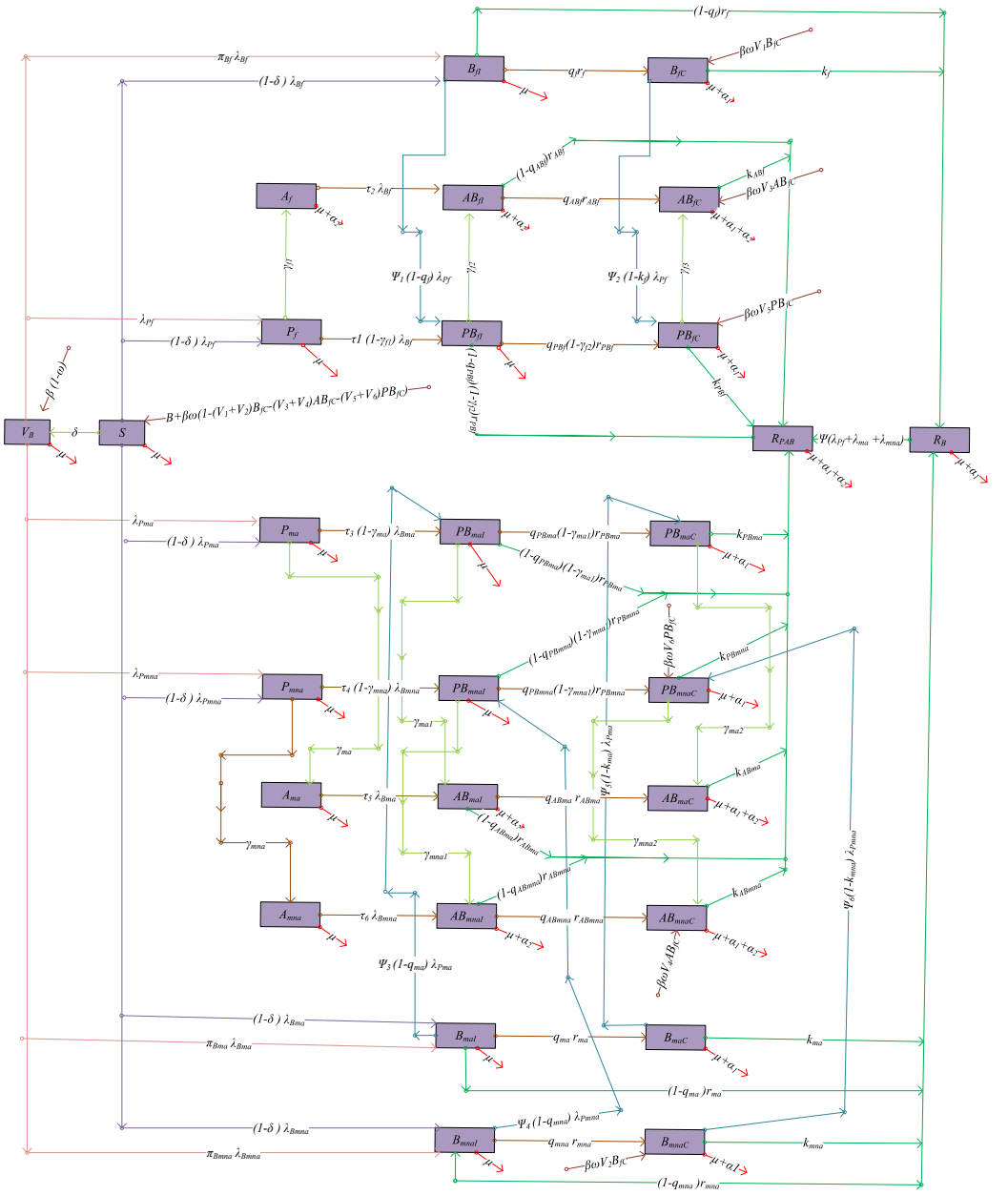

Figure 1: Disease dynamics in population.

$$\frac{dS}{dt} = B + \beta\omega\left[1 - (v_1 + v_2)B_{fC} - (v_3 + v_4)\,AB_{fC} - (v_5 + v_6)\,PB_{fC}\right] - (\delta + \mu)\,S$$
$$- (1 - \delta)\,(\lambda_{Pf} + \lambda_{pma} + \lambda_{Pmna} + \lambda_{Bf} + \lambda_{Bma} + \lambda_{Bmna})\,S - \lambda_B S - \lambda_P S, \tag{1}$$

$$\frac{dV_B}{dt} = \beta\,(1 - \omega) + \delta S - (\lambda_{pf} + \lambda_{pma} + \lambda_{pmna} + \pi_{Bf}\lambda_{Bf} + \pi_{Bma}\lambda_{Bma})\,V_B$$
$$- (\pi_{Bmna}\lambda_{Bmna})\,V_B - \mu V_B, \tag{2}$$

$$\frac{dB_{fI}}{dt} = (1 - \delta)\,\lambda_{Bf} S + \pi_{Bf}\lambda_{Bf} V_B - \psi_1\,(1 - q_f)\,\lambda_{Pf} B_{fI} - q_f r_f B_{fI} - (1 - q_f)\,r_f B_{fI}$$
$$- \mu B_{fI} + u_1\lambda_B S, \tag{3}$$

$$\frac{dB_{fC}}{dt} = \beta\omega v_1 B_{fC} - \psi_2\,(1 - k_f)\,\lambda_{pf} B_{fC} + q_f r_f B_{fI} - k_f B_{fC} - (\mu + \alpha_1)\,B_{fC}, \tag{4}$$

$$\frac{dB_{maI}}{dt} = (1 - \delta)\,\lambda_{Bma} S + \pi_{Bma}\lambda_{Bma} V_B - \psi_3\,(1 - q_{ma})\,\lambda_{pma} B_{maI} - \mu B_{maI} + u_2\lambda_B S$$
$$- q_{ma} r_{ma} B_{maI} - (1 - q_{ma})\,r_{ma} B_{maI}, \tag{5}$$

$$\frac{dB_{mnaI}}{dt} = (1 - \delta)\,\lambda_{Bmna} S + \pi_{Bmna}\lambda_{Bmna} V_B - \psi_4\,(1 - q_{mna})\,\lambda_{pmna} B_{mnaI}$$
$$- q_{mna} r_{mna} B_{mnaI} - (1 - q_{mna})\,r_{mna} B_{mnaI} + (1 - u_1 - u_2)\lambda_B S - \mu B_{mnaI}, \tag{6}$$

$$\frac{dB_{mac}}{dt} = -\psi_5\,(1 - k_{ma})\,\lambda_{pma} B_{maC} + q_{ma} r_{ma} B_{maI} - (\mu + \alpha_1)\,B_{maC} - k_{ma} B_{maC} \tag{7}$$

$$\frac{dB_{mnac}}{dt} = \beta\omega v_2 B_{fC} - \psi_6\,(1 - k_{mna})\,\lambda_{pmna} B_{mnaC} + q_{mna} r_{mna} B_{mnaI} - (\mu + \alpha_1)\,B_{mnaC}$$
$$- k_{mna} B_{mnaC}, \tag{8}$$

$$\frac{dR_B}{dt} = (1 - q)\,r_f B_{fI} + k_f B_{fC} + (1 - q_{ma})\,r_{ma} B_{maI} + k_{ma} B_{maC} - \mu R_B$$
$$+ (1 - q_{mna})\,r_{mna} B_{mnaI} + k_{mna} B_{mnaC} - \psi\,(\lambda_{pf} + \lambda_{pma} + \lambda_{pmna})\,R_B - \alpha_1 R_B, \tag{9}$$

$$\frac{dP_f}{dt} = ((1 - \delta)\,S + V_B)\,\lambda_{pf} - \tau_1\,(1 - \gamma_{f1})\,\lambda_{Bf} P_f - \mu P_f - \gamma_{f1} P_f + a_1\lambda_P S, \tag{10}$$

$$\frac{dA_f}{dt} = \gamma_{f1} P_f - \tau_2\lambda_{Bf} A_f - \mu A_f - \alpha_2 A_f, \tag{11}$$

$$\frac{dPB_{fI}}{dt} = \tau_1\,(1 - \gamma_{f1})\,\lambda_{Bf} P_f + \psi_1\,(1 - q_f)\,\lambda_{pf} B_{fI} - \mu PB_{fI} - \gamma_{f2} PB_{fI}$$
$$- q_{PBf}\,(1 - \gamma_{f2})\,r_{PBf} PB_{fI} - (1 - q_{PBf})\,(1 - \gamma_{f2})\,r_{PBf} PB_{fI}, \tag{12}$$

$$\frac{dAB_{fI}}{dt} = \gamma_{f2} PB_{fI} + \tau_2 \lambda_{Bf} A_f - (\mu + \alpha_2) AB_{fI} - q_{ABf} r_{ABf} AB_{fI} - (1 - q_{ABf}) r_{ABf} AB_{fI}, \tag{13}$$

$$\frac{dPB_{fC}}{dt} = \beta \omega v_5 PB_{fc} + q_{PBf} (1 - \gamma_{f2}) PB_{fI} r_{PBf} + \psi_2 (1 - k_f) \lambda_{pf} B_{fc} - \gamma_{f3} PB_{fc} - (\mu + \alpha_1) PB_{fc} - k_{PBf} PB_{fc}, \tag{14}$$

$$\frac{dAB_{fC}}{dt} = \beta \omega v_3 AB_{fc} + \gamma_{f3} PB_{fc} + q_{ABf} AB_{fI} r_{ABf} - (\mu + \alpha_1 + \alpha_2) PB_{fc} - k_{ABf} AB_{fc}, \tag{15}$$

$$\frac{dP_{ma}}{dt} = [(1 - \delta) S + V_B] \lambda_{pma} - \tau_3 (1 - \gamma_{ma}) \lambda_{Bma} P_{ma} - \mu P_{ma} - \gamma_{ma} P_{ma} + a_2 \lambda_P S, \tag{16}$$

$$\frac{dA_{ma}}{dt} = \gamma_{ma} P_{ma} - \tau_5 \lambda_{Bma} A_{ma} - (\mu + \alpha_2) A_{ma}, \tag{17}$$

$$\frac{dP_{mna}}{dt} = [(1 - \delta) S + V_B] \lambda_{pmna} - \tau_4 (1 - \gamma_{mna}) \lambda_{Bmna} P_{mna} - \mu P_{mna} - \gamma_{mna} P_{mna}, + (1 - a_1 - a_2) \lambda_P S, \tag{18}$$

$$\frac{dA_{mna}}{dt} = \gamma_{mna} P_{mna} - \tau_6 \lambda_{Bmna} A_{mmna} - (\mu + \alpha_2) A_{mna}, \tag{19}$$

$$\frac{dPB_{maI}}{dt} = \tau_3 (1 - \gamma_m) \lambda_{Bma} P_m + \psi_3 (1 - q_{ma}) \lambda_{pma} B_{maI} - \gamma_{ma1} PB_{maI} - q_{PBmaI} (1 - \gamma_{ma1}) r_{PBma} PB_{maI} - \mu PB_{maI} - (1 - q_{PBma}) (1 - \gamma_{ma1}) r_{PBmaI} PB_{maI}, \tag{20}$$

$$\frac{dPB_{mnaI}}{dt} = \tau_4 (1 - \gamma_m) \lambda_{Bmna} P_m + \psi_4 (1 - q_{mna}) \lambda_{pmna} B_{mnaI} - \mu PB_{mnaI} - q_{PBmnaI} (1 - \gamma_{mna1}) r_{PBmna} PB_{mnaI} - \gamma_{mna1} PB_{mnaI} - (1 - q_{PBmna}) (1 - \gamma_{mna1}) r_{PBmnaI} PB_{mnaI}, \tag{21}$$

$$\frac{dAB_{maI}}{dt} = \tau_5 \lambda_{Bma} A_m + \gamma_{ma1} PB_{maI} - q_{ABma} r_{ABma} AB_{maI} - (1 - q_{ABma}) r_{ABma} AB_{maI} - (\mu + \alpha_2) AB_{maI}, \tag{22}$$

$$\frac{dAB_{mnaI}}{dt} = \tau_6 \lambda_{Bmna} A_m + \gamma_{mna1} PB_{mnaI} - q_{ABmna} r_{ABmna} AB_{mnaI} - (1 - q_{ABmna}) r_{ABmna} AB_{mnaI} - (\mu + \alpha_2) AB_{mnaI}, \tag{23}$$

$$\frac{dPB_{mac}}{dt} = q_{PBma}\left(1-\gamma_{ma1}\right)r_{PBma}PB_{maI} + \psi_5\left(1-k_{ma}\right)\lambda_{Pma}B_{mac} - k_{PBma}PB_{mac} - \gamma_{ma2}PB_{mac} - \left(\mu+\alpha_1\right)PB_{mac}, \tag{24}$$

$$\frac{dPB_{mnac}}{dt} = \beta\omega v_6 PB_{fc} + q_{PBmna}\left(1-\gamma_{mna1}\right)r_{PBmna}PB_{mnaI} + \psi_6\left(1-k_{mna}\right)\lambda_{Pmna}B_{mnac} - k_{PBmna}PB_{mnac} - \gamma_{mna2}PB_{mnac} - \left(\mu+\alpha_1\right)PB_{mnac}, \tag{25}$$

$$\frac{dAB_{mac}}{dt} = \gamma_{ma2}PB_{mac} + q_{ABma}r_{ABma}AB_{maI} - k_{ABma}AB_{mac} - \left(\mu+\alpha_1+\alpha_2\right)AB_{mnac}, \tag{26}$$

$$\frac{dAB_{mnac}}{dt} = \beta\omega v_4 AB_{mac} + \gamma_{mna2}PB_{mnac} + q_{ABmna}r_{ABmna}AB_{mnaI} - k_{ABmna}AB_{mnac} - \left(\mu+\alpha_1+\alpha_2\right)AB_{mnac}, \tag{27}$$

$$\begin{aligned}\frac{dR_{PAB}}{dt} &= \psi\left(\lambda_{Pf}+\lambda_{ma}+\lambda_{mna}\right)R_{PAB} + \left(1-q_{PBf}\right)\left(1-\gamma_{f2}\right)r_{PBf}PB_{fI} \\ &+ \left(1-q_{ABf}\right)r_{ABf}AB_{fI} + \left(1-q_{ABma}\right)r_{ABma}AB_{maI} + \left(1-q_{ABmna}\right)r_{ABmna}AB_{mnaI} \\ &+ \left(1-q_{PBmna}\right)\left(1-\gamma_{mna1}\right)r_{PBmna}PB_{mnaI} + \left(1-q_{PBma}\right)\left(1-\gamma_{ma1}\right)r_{PBma}PB_{maI} \\ &+ k_{ABf}AB_{fc} + k_{PBf}PB_{fc} + k_{ABma}AB_{mac} + k_{PBma}PB_{mac} \\ &+ k_{PBmna}PB_{mnac} + k_{ABmna}AB_{mnac} - \left(\mu+\alpha_1+\alpha_2\right)R_{PAB},\end{aligned} \tag{28}$$

with,

$$\begin{aligned}N(t) &= S(t) + V_B(t) + B_{fI}(t) + B_{fC}(t) + B_{maI}(t) + B_{mnaI}(t) + B_{maC}(t) + B_{mnaC}(t) \\ &+ R_B(t) + R_{PAB}(t) + P_f(t) + A_f(t) + P_{ma}(t) + A_{ma}(t) + P_{mna}(t) + A_{mna}(t) \\ &+ PB_{fI}(t) + AB_{fI}(t) + PB_{fC}(t) + AB_{fC}(t) + PB_{maI}(t) + AB_{maI}(t) + PB_{mnaI}(t) \\ &+ AB_{mnaI}(t) + PB_{maC}(t) + AB_{maC}(t) + PB_{mnaC}(t) + AB_{mnaC}(t),\end{aligned} \tag{29}$$

where force of infections are defined by

$$\begin{aligned}\lambda_{Bf} &= \beta_{Bf}[(B_{maI}+B_{mnaI}) + \omega_1((PB_{maI}+PB_{mnaI}) + \omega_2(AB_{maI}+AB_{mnaI})) \\ &+ \omega_3(((B_{maC}+B_{mnaC}) + \omega_4(PB_{maC}+PB_{mnaC})) + \omega_5(AB_{maC}+AB_{mnaC}))],\end{aligned}$$

$$\lambda_{Bma} = \beta_{Bma}[B_{fI} + \omega_6(PB_{fI} + \omega_7 AB_{fI}) + \omega_8(B_{fC} + \omega_9(PB_{fC} + \omega_{10}AB_{fC}))],$$

$$\lambda_{Bmna} = \beta_{Bmna}[B_{fI} + \omega_{11}(PB_{fI} + \omega_{12}AB_{fI}) + \omega_{13}(B_{fC} + \omega_{14}(PB_{fC} + \omega_{15}AB_{fC}))],$$

$$\begin{aligned}\lambda_{Pf} &= \beta_{Pf}[(P_{ma}+P_{mna}+\eta_1(A_{ma}+A_{mna})) + \eta_2(PB_{maI}+PB_{mnaI}+\eta_3(AB_{maI}+AB_{mnaI})) \\ &+ \eta_4(PB_{maC}+PB_{mnaC}+\eta_5(AB_{maC}+AB_{mnaC}))],\end{aligned}$$

$$\lambda_{Pma} = \beta_{Pma}[(P_f + \eta_6 A_f) + \eta_7(PB_{fI} + \eta_8 AB_{fI}) + \eta_9(PB_{fC} + \eta_{10}AB_{fC})],$$

$$\lambda_{Pmna} = \beta_{Pmna}[(P_f + \eta_{11}A_f) + \eta_{12}(PB_{fI} + \eta_{13}AB_{fI}) + \eta_{14}(PB_{fC} + \eta_{15}AB_{fC})],$$

$$\lambda_B = \beta_B[(B_{fI} + B_{maI} + B_{mnaI}) + \omega_{16}(B_{fC} + B_{maC} + B_{mnaC}) + \omega_{17}(PB_{fI} + PB_{maI} + PB_{mnaI} + AB_{fI} + AB_{maI} + AB_{mnaI} + PB_{fC} + PB_{maC} + PB_{mnaC} + AB_{fC} + AB_{maC} + AB_{mnaC})],$$

$$\lambda_P = \beta_P[(P_f + P_{ma} + P_{mna}) + \eta_{16}(A_f + A_{ma} + A_{mna}) + \eta_{17}(PB_{fI} + PB_{maI} + PB_{mnaI} + AB_{fI} + AB_{maI} + AB_{mnaI} + PB_{fC} + PB_{maC} + PB_{mnaC} + AB_{fC} + AB_{maC} + AB_{mnaC})].$$

The feasible solution of system of equations given by (1) to (8) and (10) to (27) is in the region

$$\Omega, = \left\{ \left(\begin{array}{l} S, V_B, B_{fI}, B_{fC}, B_{maI}, B_{maC}, B_{mnaI}, B_{mnaC}, P_f, A_f, PB_{fI}, \\ AB_{fI}, PB_{fC}, AB_{fC}, P_{ma}, P_{mna}, A_{ma}, A_{mna}, PB_{maI}, PB_{mnaI}, \\ AB_{maI}, AB_{mnaI}, PB_{maC}, PB_{mnaC}, AB_{maC}, AB_{mnaC} \end{array} \right) \geq 0, N(t) \leq \frac{B+\beta}{\mu} \right\}.$$

3 Stability Analysis

3.1 Disease free Equilibrium (DFE)

The HIV-HBV model (1) to (28) has a DFE

$$X_0 = \left(\begin{array}{l} \frac{B+\beta\omega}{\mu+\delta}, + \frac{\mu\beta(1-\omega)+\delta(B+\beta)}{\mu(\mu+\delta)}, 0, 0, 0, 0, 0, 0, 0, 0, 0, \\ 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0 \end{array} \right).$$

Basic Reproduction number at DFE

Model for HIV only

Using equations (10), (11),(16),(17),(18) and (19) with next generation matrix method proposed by Driessche and Watmough, basic reproduction number for HIV only is derived.

Let

$$X_H' = (P_f, A_f, P_{ma}, A_{ma}, P_{mna}, A_{mna})',$$

where dash denotes derivative.

$$\therefore \quad X_H' = \frac{dX_H}{dt} = \Im_H(X_H) - v_H(X_H),$$

where,

$$\Im_H(X_H) = \begin{bmatrix} ((1-\delta)S + V)\beta_{pf}[(P_{ma} + P_{mna}) + \eta_1(A_{ma} + A_{mna})] + a_1\beta_P[(P_f + P_{ma} + P_{mna}) + \eta_{16}(A_f + A_{ma} + A_{mna})]S \\ ((1-\delta)S + V)\beta_{Pma}(P_f + \eta_6 A_f) + a_2\beta_P[(P_f + P_{ma} + P_{mna}) + \eta_{16}(A_f + A_{ma} + A_{mna})]S \\ ((1-\delta)S + V)\beta_{Pmna}(P_f + \eta_6 A_f) + a_2\beta_P[(P_f + P_{ma} + P_{mna}) + \eta_{16}(A_f + A_{ma} + A_{mna})]S \\ 0 \\ 0 \\ 0 \end{bmatrix},$$

and,

$$\upsilon_H\left(X_H\right)=\begin{bmatrix}\left(\mu+\gamma_{f1}\right)P_f\\ \left(\mu+\gamma_{ma}+\alpha_2\right)P_{ma}\\ \left(\mu+\gamma_{mna}+\alpha_2\right)P_{mna}\\ \left(\mu+\alpha_2\right)A_f-\gamma_{f1}P_f\\ \left(\mu+\alpha_2\right)A_{ma}-\gamma_{ma}P_{ma}\\ \left(\mu+\alpha_2\right)A_{mna}-\gamma_{mna}P_{mna}\end{bmatrix}.$$

Using $F_1=\left[\frac{\partial \Im_{Hi}(X_0)}{\partial X_j}\right]$ and $V_1=\left[\frac{\partial \upsilon_{Hi}(X_0)}{\partial X_j}\right]$ for $i,j=1,2,3,...6$,

$$F_1=\begin{bmatrix}A_{11} & A_{11}+A_{12} & A_{11}+A_{12} & \eta_{16}A_{11} & \eta_{16}A_{11}+\eta_1A_{12} & \eta_{16}A_{11}+\eta_1A_{12}\\ A_{21}+A_{22} & A_{21} & A_{21} & \eta_{16}A_{21}+\eta_6A_{22} & \eta_{16}A_{21} & \eta_{16}A_{21}\\ A_{31}+A_{32} & A_{31} & A_{31} & \eta_{16}A_{31}+\eta_6A_{32} & \eta_{16}A_{31} & \eta_{16}A_{31}\\ 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0\end{bmatrix},$$

where,

$$\begin{aligned} A_{11}&=a_1\beta_PS, & A_{12}&=\left((1-\delta)S+V\right)\beta_{pf},\\ A_{21}&=a_2\beta_PS, & A_{12}&=\left((1-\delta)S+V\right)\beta_{pma},\\ A_{31}&=(1-a_1-a_2)\beta_PS, & A_{32}&=\left((1-\delta)S+V\right)\beta_{pmna},\end{aligned}$$

and

$$V_1=\begin{bmatrix}\mu+\gamma_{f1} & 0 & 0 & 0 & 0 & 0\\ 0 & \mu+\alpha_2+\gamma_{ma} & 0 & 0 & 0 & 0\\ 0 & 0 & \mu+\alpha_2+\gamma_{mna} & 0 & 0 & 0\\ -\gamma_{f1} & 0 & 0 & \mu+\alpha_2 & 0 & 0\\ 0 & -\gamma_{ma} & 0 & 0 & \mu+\alpha_2 & 0\\ 0 & 0 & -\gamma_{mna} & 0 & 0 & \mu+\alpha_2\end{bmatrix}.$$

Basic reproduction number R_0^H for HIV only is largest Eigenvalue of $F_1V_1^{-1}$, with the parametric values given in Table 1 at disease free equilibrium, $R_0^H=0.6133$.

Model for HBV only

Using equations (1) to (8), as discussed above the basic reproduction number can be computed as follows. Let

$$X'_B=\left(B_{fI},B_{fc},B_{maI},B_{mac},B_{mnaI},B_{mnac},S_B,V_B\right)',$$

where dash denotes derivative.

$$\therefore X'_B=\frac{dX_B}{dt}=\Im_B\left(X_B\right)-\upsilon_B\left(X_B\right),$$

where,

$$\Im_B(X_B) = \begin{bmatrix} \lambda_{Bf}\left[\pi_{bf}V_B + (1-\delta)S_B\right] + u_1[B_{fI} + B_{maI} + B_{mnaI} + \omega_{16}(B_{fC} + B_{maC} + B_{mnaC})]S_B \\ \lambda_{Bma}\left[\pi_{bma}V_B + (1-\delta)S_B\right] + u_2[B_{fI} + B_{maI} + B_{mnaI} + \omega_{16}(B_{fC} + B_{maC} + B_{mnaC})]S_B \\ \lambda_{Bmna}\left[\pi_{bmna}V_B + (1-\delta)S_B\right] + (1-u_1-u_2)[B_{fI} + B_{maI} + B_{mnaI} + \omega_{16}(B_{fC} + B_{maC} + B_{mnaC})]S_B \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix},$$

and

$$\upsilon(X_B) = \begin{bmatrix} (\mu + r_f)B_{fI} \\ (\mu + r_{ma})B_{maI} \\ (\mu + r_{mna})B_{mnaI} \\ -q_f r_f B_{fI} + (\mu + \alpha_1 + k_f - \beta\omega v_1)B_{fc} \\ -q_{ma} r_{ma} B_{maI} + (\mu + \alpha_1 + k_{ma})B_{mac} \\ -q_{mna} r_{mna} B_{mnaI} + (\mu + \alpha_1 + k_{mna})B_{mnac} - \beta\omega v_2 B_{fc} \\ -\beta(1-\omega) - \delta S_B + (\pi_{bf}\lambda_{Bf} + \pi_{bma}\lambda_{Bma} + \pi_{bmna}\lambda_{Bmna} + \mu)V_B \\ -B - \beta\omega + (1-\delta)(\lambda_{Bf} + \lambda_{Bma} + \lambda_{Bmna})S_B + \lambda_B S_B + (\mu + \delta)S_B \end{bmatrix}.$$

Using $\left[\frac{\partial \Im_{Bi}(X_0)}{\partial X_j}\right] = \begin{bmatrix} F_2 & 0 \\ 0 & 0 \end{bmatrix}$ and $\left[\frac{\partial \upsilon_{Bi}(X_0)}{\partial X_j}\right] = \begin{bmatrix} V_2 & 0 \\ J_1 & J_2 \end{bmatrix}$ for $i, j = 1, 2, 3, ...8$,

$$F_2 = \begin{bmatrix} B_{11} & B_{11}+B_{12} & lB_{11}+B_{12} & \omega_{16}B_{11} & \omega_{16}B_{11}+\omega_3 B_{12} & \omega_{16}B_{11}+\omega_3 B_{12} & 0 & 0 \\ B_{21}+B_{22} & B_{21} & B_{21} & \omega_{16}B_{21}+\omega_3 B_{22} & \omega_{16}B_{21} & \omega_{16}B_{21} & 0 & 0 \\ B_{31}+B_{32} & B_{31} & B_{31} & \omega_{16}B_{31}+\omega_3 B_{32} & \omega_{16}B_{31} & \omega_{16}B_{31} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

where,

$$\begin{aligned} B_{11} &= u_1\beta_B S_B, & B_{12} &= \beta_{bf}\left(\pi_{bf}V_B + (1-\delta)S_B\right), \\ B_{21} &= u_2\beta_B S_B, & B_{22} &= \beta_{ma}\left(\pi_{bma}V_B + (1-\delta)S_B\right), \\ B_{31} &= (1-u_1-u_2)\beta_B S_B, & B_{32} &= \beta_{mna}\left(\pi_{bmna}V_B + (1-\delta)S_B\right). \end{aligned}$$

$$V_2 =$$

$$\begin{bmatrix} \mu+r_f & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \mu+r_{ma} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \mu+r_{mna} & 0 & 0 & 0 & 0 & 0 \\ -q_f r_{-f} & 0 & 0 & -\beta\omega v_1+\mu+\alpha_1+k_f & 0 & 0 & 0 & 0 \\ 0 & -q_{ma} r_{ma} & 0 & 0 & \mu+\alpha_1+k_{ma} & 0 & 0 & 0 \\ 0 & 0 & -q_{mna}\gamma_{mna} & -\beta\omega v_2 & 0 & \mu+\alpha_1+k_{mna} & 0 & 0 \\ B_{71}+B_{72} & B_{73} & B_{73} & \omega_8 B_{71}+\omega_{13} B_{72} & \omega_3 B_{73} & \omega_3 B_{73} & \mu & -\delta \\ B_{81}+B_{82}+B_\beta & B_{83}+B_\beta & B_{83}+B_\beta & \omega_8 B_{81}+\omega_{13} B_{82}+\omega_{16} B_\beta & \omega_3 B_{83}+\omega_{16} B_\beta & \omega_3\,(1-\delta)\,S\beta_{bf} & 0 & \mu+\delta \end{bmatrix},$$

where,

$$\begin{array}{lll} B_{71}=\pi_{Bma}\beta_{Bma}, & B_{72}=\pi_{Bmna}\beta_{Bmna}, & B_{73}=\pi_{Bf}\beta_{Bf}, \\ B_{81}=(1-\delta)\beta_{Bma}, & B_{82}=(1-\delta)\beta_{Bmna}, & B_{83}=\ (1-\delta)\beta_{bf}. \end{array}$$

Basic reproduction number $R_0^B$for HBV only is largest Eigenvalue of $F_2V_2^{-1}$with the given parametric values in Table 1 and at disease free equilibrium $R_0^B = 0.1592$.

HIV-HBV co-infection at DFE

Hence, $R_B^H = R_0^H\left(1-\frac{\delta S}{N}\right)$= 0.6013.

3.2 Existence and Stability of Boundary Equilibria

The HBV only (HIV free) equilibrium X_B is given by

$$S_B = \frac{B+\beta\omega(1-(v_1+v_2)\,B_{fC})}{\delta+\mu+(1-\delta)(\lambda_{Bf}+\lambda_{Bma}+\lambda_{Bmna})+\lambda_B},$$

$$V_B = \frac{\beta(1-\omega)+\delta S_B}{\pi_{Bf}\lambda_{Bf}+\pi_{Bma}\lambda_{Bma}+\pi_{Bmna}\lambda_{Bmna}+\mu},$$

$$B_{fI} = \frac{(1-\delta)\lambda_{Bf}S_B+\pi_{Bf}V_B+u_1\lambda_B S_B}{r_f+\mu},$$

$$B_{maI} = \frac{(1-\delta)\lambda_{Bma}S_B+\pi_{Bma}\lambda_{Bma}V_B+u_2\lambda_B S_B}{r_{ma}+\mu},$$

$$B_{mnaI} = \frac{(1-\delta)\lambda_{Bmna}S_B+\pi_{Bmna}\lambda_{Bmna}V_B+(1-u_1-u_2)\lambda_B S_B}{r_{mna}+\mu},$$

$$B_{fC} = \frac{q_f r_f B_{fI}}{k_f+\mu+\alpha_1-\beta\omega v_1},$$

$$B_{maC} = \frac{q_{ma}r_{ma}B_{maI}}{k_{ma}+\mu+\alpha_1},$$

$$B_{mnaC} = \frac{\beta\omega v_2 B_{fC}+q_{mna}r_{mna}B_{mnaI}}{k_{mna}+\mu+\alpha_1},$$

$$\lambda_{Bf} = \beta_{Bf}[(B_{maI}+B_{mnaI})+\omega_1((PB_{maI}+PB_{mnaI}))],$$

$$\lambda_{Bma} = \beta_{Bma}[B_{fI}+\omega_8 B_{fC}],$$

$$\lambda_{Bmna} = \beta_{Bmna}[B_{fI}+\omega_{13}B_{fC}],$$

$$\lambda_B = \beta_B[(B_{fI}+B_{maI}+B_{mnaI})+\omega_{16}(B_{fC}+B_{maC}+B_{mnaC})].$$

To obtain the capacity of HIV to invade and persists in a population when HBV is at equilibrium

Using equations (10),(11), (16), (17), (18) and (19), we compute reproduction number at X_B.

Let

$$X'^{H}{}_{B} = (P_f, A_f, P_{ma}, A_{ma}, P_{mna}, A_{mna})',$$

where dash denotes derivative.

$$X'^{H}{}_{B} = \frac{dX_B{}^{H}}{dt} = \Im(X) - \upsilon(X),$$

where,

$$\Im(X) = \Im_H(X_H),$$

and

$$\upsilon(X) = \begin{bmatrix} (\mu + \gamma_{f1} + \tau_1(1-\gamma_{f1})\lambda_{Bf})P_f \\ (\mu + \gamma_{ma} + \alpha_2 + \tau_3(1-\gamma_{ma})\lambda_{Bma})P_{ma} \\ (\mu + \gamma_{mna} + \alpha_2 + \tau_4(1-\gamma_{mna})\lambda_{Bmna})P_{mna} \\ (\mu + \alpha_2 + \tau_2\lambda_{Bf})A_f - \gamma_f P_f \\ (\mu + \alpha_2 + \tau_5\lambda_{Bma})A_{ma} - \gamma_{ma}P_{ma} \\ (\mu + \alpha_2 + \tau_6\lambda_{Bmna})A_{mna} - \gamma_{mna}P_{mna} \end{bmatrix}.$$

Using $F_3 = \left[\frac{\partial \Im_i(X_0)}{\partial X_j}\right]$ and $V_3 = \left[\frac{\partial \upsilon_i(X_0)}{\partial X_j}\right]$ for $i, j = 1, 2, 3, ..., 6$, we have, $F_3 = F_1$ and

$$V_3 = \begin{bmatrix} V_{11} & 0 & 0 & 0 & 0 & 0 \\ 0 & V_{22} & 0 & 0 & 0 & 0 \\ 0 & 0 & V_{33} & 0 & 0 & 0 \\ -\gamma_{f1} & 0 & 0 & V_{44} & 0 & 0 \\ 0 & -\gamma_{ma} & 0 & 0 & V_{55} & 0 \\ 0 & 0 & -\gamma_{mna} & 0 & 0 & V_{66} \end{bmatrix}.$$

where,

$$\begin{aligned} V_{11} &= \mu + \gamma_{f1} + \tau_1(1-\gamma_{f1})\lambda_{Bf}, \\ V_{22} &= \mu + \alpha_2 + \gamma_{ma} + \tau_3(1-\gamma_{ma})\lambda_{Bma}, \\ V_{33} &= \mu + \alpha_2 + \gamma_{mna} + \tau_4(1-\gamma_{mna})\lambda_{Bmna}, \\ V_{44} &= \mu + \alpha_2 + \tau_2\lambda_{Bf}, \\ V_{55} &= \mu + \alpha_2 + \tau_5\lambda_{Bma}, \\ V_{66} &= \mu + \alpha_2 + \tau_6\lambda_{Bmna}. \end{aligned}$$

Basic reproduction number $R_B^H$for HBV in population where HIV is fixed is largest eigenvalue of $F_3V_3^{-1}$ at X_B and with the given parameter values$R_B^H(X_B) = 0.6133$. This is due to vaccination.

4 Simulation and Numerical analysis

Now we carry out simulation to study stability of the system which indicates the nature of the co-infection.

Table 2: Effect of change in model parameters on stability.

Increase in model parameters	System stability	Increase in model parameters	System stability
β	Unstable	B	Unstable
β_P	Unstable	μ	Stable
β_{Pf}	Unstable	δ	Stable
β_{Pma}	Unstable	α_2	Stable
β_{Pmna}	Unstable		

From Table 2, it can be observed that increase in number of susceptibles, rate of new birth, rate of HIV infection sexual and nonsexual makes system unstable. While, natural death, vaccination rate, HIV induced death rate makes system stable. Also, it is observed that increase in birth rate and decrease in vaccination rate simultaneously moves system faster towards unstability.

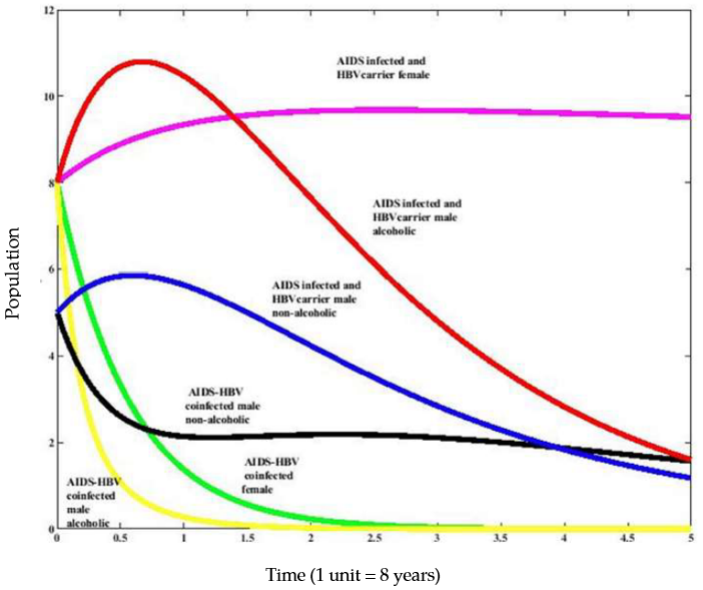

Figure 2: HIV - HBV co-infection with liquor habit.

From Figure 2, it can be observed that the alcoholic habits increases AIDS infected HBV carrier alcoholic male individuals and result of it is decrease in AIDS-HBV co-infected individuals. Also, alcoholic male individuals move faster towards carrier class then non- alcoholic males. As female HIV- HBV co-infected are at high risk of moving towards carrier but effect of vaccination makes HIV infected HBV carrier female class stable after time period.

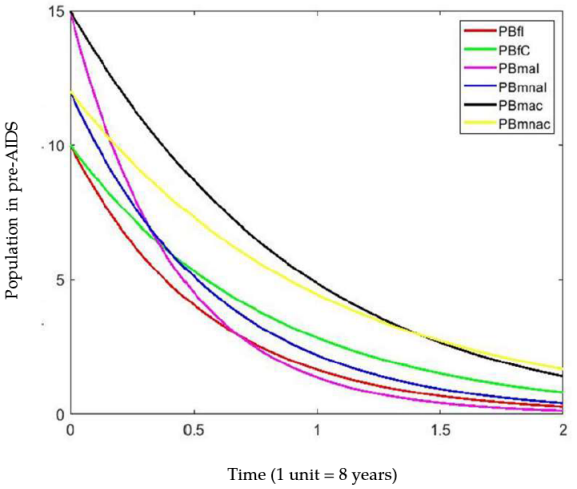

Figure 3: Effect of vaccination on pre-AIDS co-infected classes.

In Figure3, effects of vaccination on Pre-AIDS HBV co-infected and carrier classes are shown. It shows as time increases infected moves faster towards stability then carrier classes.

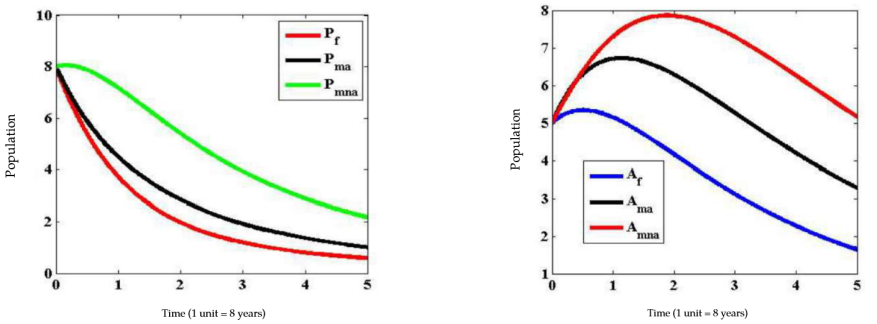

Figure 4: Dynamics of pre-AIDS and AIDS classes.

In Figure 4, dynamics of Pre-AIDS and AIDS classes are obtained. It shows as AIDS individuals increases faster than the pre- AIDS individuals initially but thereafter all moves towards stability. Also it can be observed that liquor habit does not affect AIDS spread.

In Figure 5, dynamics of HBV amongst various classes is shown and it indicates that number of alcoholic HBV carrier male individuals increases faster than number of non-alcoholic HBV carrier male individuals and number of alcoholic HBV carrier female individuals. Also it can be clearly observed that liquor habit affect HBV spread.

Figure 6 indicates that HIV-HBV coinfected individuals move faster towards recover class than only HBV infected individuals initially from HBV infection.

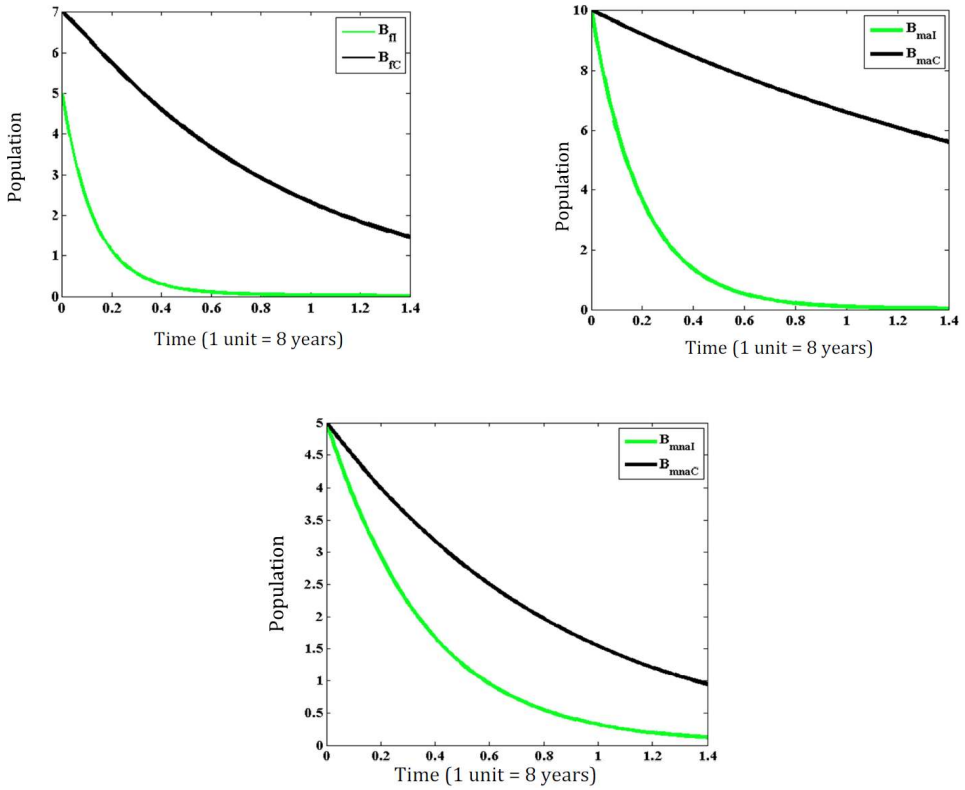

Figure 5: Dynamics of HBV amongst various population.

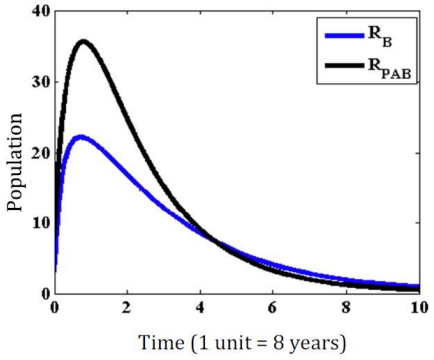

Figure 6: Dynamics of HBV - recovered individuals.

From Figure 7, it can be observed that the susceptible increases initially and then moves toward stability while vaccinated individuals increase as higher vaccination policy is applied to control disease spread.

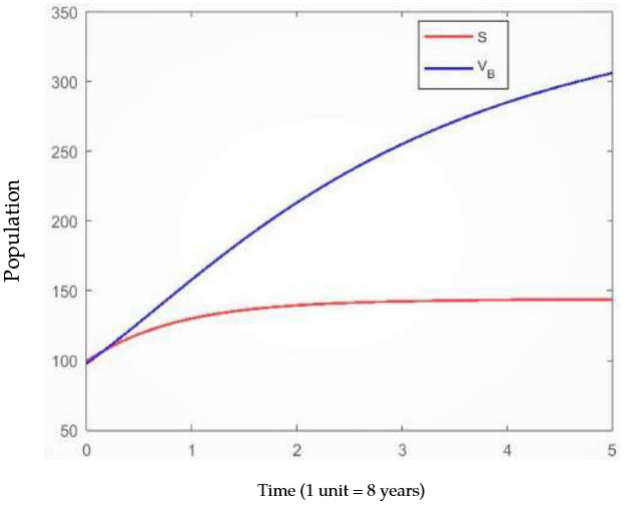

Figure 7: Dynamics of susceptible and vaccinated classes.

5 Conclusion

In this paper, HIV-HBV co-infection is discussed. From the next generation matrix the basic reproduction number for the HIV model, HBV model and HIV-HBV co-infected model are computed. Stability at the equilibrium states for model parameters is carried out and numerical simulation is also worked out. The effects of alcoholic habits, new birth without vaccination and with vaccination on endemic behaviour of the model are investigated. The more vaccination amongst new birth is suggested to keep disease spread in control. The awareness campaigns about sexual, non-sexual transmission of HIV and risk of alcoholic habits should be organised.

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Conflicts of Interest The authors declare no conflict of interest.

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